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TURBULENT CONVECTION: OLD AND NEW MODELS

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ABSTRACT

This paper contains (1) a physical argument to show that the one-eddy MLT model *underestimates* the convective flux F_c in the high-efficiency regime, while it *overestimates* F_c in the low-efficiency regime, and (2) a new derivation of the F_c(MLT) using a turbulence model in the one-eddy approximation. (3) We forsake the one-eddy approximation and adopt the Kolmogorov spectrum to represent the turbulent energy spectrum. The resulting $F_c > F_c$(MLT) in the high-efficiency regime, and $F_c < F_c$(MLT) in the low-efficiency case, are in agreement with the physical arguments concerning the one-eddy MLT model. (4) By forsaking the Kolmogorov model and solving a two-point closure model, one obtains the CM model. The F_c(CM) satisfies (1). F_c(CM) corresponds to a "tilt" in efficiency space of F_c(MLT), an effect that cannot be achieved by changing α. We discuss the astrophysical tests of the CM model.

(5) Concerning the laboratory turbulent convection, we show that the CM model provides a better fit than the MLT to recent high Rayleigh number (Ra) laboratory data on convection. (6) Concerning nonlocal convection, the most complete model available is the one-point closure model (Reynolds stress model), which entails five differential equations for the five second-order moments. We present the solution corresponding to the local, stationary case. The results are expressed analytically in terms of Ko (Kolmogorov constant), Pe (Peclet number), and S (convective efficiency).

(7) We find that the superadiabatic temperature gradient is given by $-\partial T/\partial r - c_p^{-1} g^r$ where the renormalized $g^r = g(1 + g^{-1}\rho^{-1} dp_t/dz)$ and p_t is the turbulent pressure. This result, which follows naturally from the Reynolds stress approach, contrasts with previous empirical suggestions to include p_t. (8) We derive new expressions for the turbulence pressure using two different turbulence models and (9) we show that the often used Kolmogorov-Prandtl expression for the turbulent diffusivity is valid only in the high convective efficiency limit. We derive a new expression valid for arbitrary Peclet numbers. (10) We derive an expression for the flux conservation law, which includes F(KE), the flux of turbulent kinetic energy, a third-order moment for which we provide a new expression. (11) No model has thus far accounted for the influence on F_c due to the presence of a stable layer (radiative layer) bordering the convective zone. We work out the first such model, and (12) we discuss topics for future research.

Subject headings: convection — stars: interiors — turbulence

1. TURBULENCE

Turbulence is not an intrinsic property of a "flow in motion." This implies that in order to be generated and maintained, turbulence requires a source of energy. We call ϵ the energy per gram, per second, needed to keep a turbulent state from decaying in time. In general, this energy is fed at the largest scales that are sensitive to the specific nature of the external force, have the longest lifetime, are affected by the geometry of the system, and are mostly diffusive rather than dissipative, that is, are unaffected by molecular processes (e.g., viscosity). By contrast, small scales (eddies) are insensitive to the specific nature of the stirring force but not to the total amount of energy fed into the system, have shorter lifetimes, are not affected by the geometry of the system, and are dissipative rather than diffusive; that is, they are affected by molecular processes. Through the action of the nonlinear terms in the Navier-Stokes equations (NSEs), the energy input ϵ is distributed, mainly cascaded, to the smaller entities, so that at any given time, small, medium, and large eddies coexist. This process of energy distribution results in an energy (density) spectrum $E(k)$, where the wavenumber k is roughly related to the inverse of the "size of the eddy." The integral of $E(k)$ over all eddy sizes yields the total turbulent kinetic energy

$$K = \tfrac{1}{2}v_t^2 = \int dk\, E(k) , \tag{1a}$$

and a major challenge of a turbulent model is the derivation of the spectral function $E(k)$.

Can $E(k)$ be represented by a δ-function, or, equivalently, how wide is the eddy spectrum? If L and l represent the largest and smallest (so defined that it is affected by dissipation), we have (see below)

$$\frac{L}{l} = \mathrm{Re}^{3/4} . \tag{1b}$$

In stellar interiors, Re $\sim 10^{10}$, the width is greater than 10^7, which can hardly be represented by a δ-function.

One of the fundamental properties of the nonlinear interactions is that they conserve energy. This means that the energy fed at the largest scales remains unchanged during the "cascade" process, with the result that ϵ must ultimately be dissipated by the physical mechanisms operating at the smallest scales; that is, molecular viscosity, which causes ϵ to be dissipated into heat. Viscosity enters the NSE as $-\nu\nabla^2 u_i$; once we multiply the NSE by u_i to obtain the energy equation, the above statement corresponds to saying that

$$\epsilon = 2\nu \int k^2 E(k) dk , \tag{2a}$$

where the k^2 factor comes from the ∇^2. Equation (2a) may lead one to conclude that when ν is exceedingly small, as in stellar interiors, ϵ is correspondingly small, or even worse, that when $\nu \to 0$, ϵ vanishes. This is an incorrect inference since equation (2a) only states that for a given value of ϵ, determined by the source, dissipation occurs predominantly at large wavenumbers, that is at the smallest scales. One can also interpret equation (2a) as a statement of

$$\text{production} = \text{dissipation} . \tag{2b}$$

The quantity multiplying 2ν is the integral of the enstrophy (mean-square vorticity),

$$\omega(k) = k^2 E(k) , \tag{2c}$$

and since this quantity increases as $\nu \to 0$, the right-hand side of equation (2a) remains finite for any ν, as expected, since the amount of energy input into a system is considered independent of the viscosity of the system.

The fact that the energy fed at the largest scales remains unchanged by the nonlinear interactions is of great relevance. It implies, among other things, that it can be used to construct a "dissipation scale" l_d characterizing the scales where dissipation occurs. Since this process occurs because of viscosity, one has only two variables at one's disposal, ϵ and ν ($\mathrm{cm}^2\ \mathrm{s}^{-1}$), and it thus follows that

$$l_d = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} . \tag{3a}$$

Since on the other hand, we also have that

$$\epsilon = U^2/T = U^3 L^{-1} , \tag{3b}$$

we derive ($\mathrm{Re} = UL\nu^{-1}$)

$$\frac{L}{l_d} = \mathrm{Re}^{3/4} , \tag{4a}$$

which is equation (1b). Thus, the smaller the viscosity, the larger the ration L/l_d. Figure 1 exhibits several illustrative cases with different viscosities; $\sigma = \nu/\chi$ is the molecular Prandtl number which for the Sun is $\sigma = 10^{-9}$ (Massaguer 1990), while for air is $\sigma = 0.7$. As one can see, the smaller the ν, the wider the spectrum width is and the less reliable a

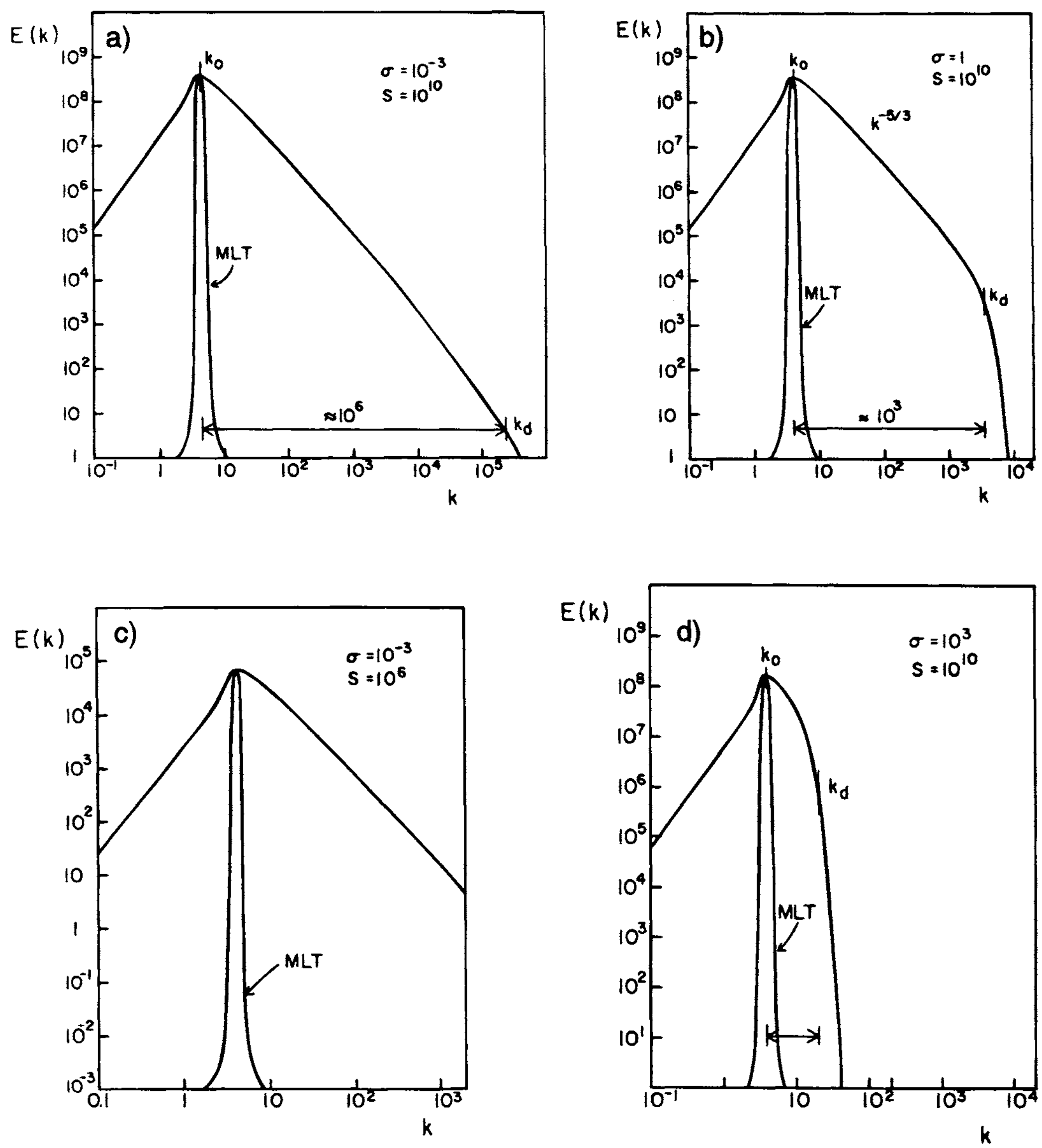

FIG. 1.—Four examples of turbulent kinetic energy spectra $E(k)$ vs. k computed with the EDQNM model. The MLT approximation consists of using a Dirac δ-function for $E(k)$. We recall that $S = \sigma$ Ra, where σ is the Prandtl number and Ra is the Rayleigh number. The value of σ in the Sun is around 10^{-10} but the results saturate at $\sigma \sim 10^{-3}$. Clearly, the higher the viscosity, the better the MLT approximation.

δ-function representation,

$$E(k) = E_0 \, \delta(k/k_0 - 1) , \quad (4b)$$

whose validity (if any) is restricted to the case of low Reynolds number and large viscosities, quite the opposite of what one encounters in stellar interiors. On this basis alone, one can justifiably have doubts about the reliability of the mixing length theory (MLT) as a faithful representation of stellar turbulent convection.

2. MIXING LENGTH THEORY

Here, we study the consequences of equation (4b). Calling H_c the convective flux F_c in units of $c_p \rho$,

$$H_c \equiv \overline{w\theta} , \quad (5a)$$

where w and θ are the fluctuating velocity (in the z-direction) and temperature fields, it is easy to show that (Yamaguchi 1963)

$$H_c = \frac{1}{g\alpha} \int_{k0}^{\infty} 2[\omega(k) + \nu k^2] E(k) dk , \quad (5b)$$

where α is the volume expansion coefficient ($= T^{-1}$ for a perfect gas) and $\omega(k)$ is the rate of energy input into the system. Thus,

$$g\alpha\overline{w\theta} = \int_{k0}^{\infty} 2[\omega(k) + \nu k^2] E(k) dk . \quad (5c)$$

Because of equations (2a)–(2b), where production $\equiv g\alpha\overline{w\theta}$ and dissipation $\equiv \epsilon$, it follows that

$$\int_{k0}^{\infty} 2\omega(k) E(k) dk = 0 , \quad (5d)$$

which is satisfied only if $\omega(k)$ has both positive (low k's) and negative components (high k's). This is indeed the case (Canuto et al. 1991, Fig. 1; Canuto & Mazzitelli 1991 [hereafter CM], eq. [18]). Let us return to equation (5b) and consider the choice of $\omega(k)$. We have two timescales:

$$t_b = (g\alpha\beta)^{-1/2} , \quad t_\chi = \frac{l^2}{\chi} \sim \frac{1}{\chi k^2} . \quad (6a)$$

The first is the buoyancy timescale (β is the superadiabatic gradient), while the second is the radiative timescale. Here, χ is the thermometric conductivity, $K/c_p \rho$, where K is the radiative conductivity. We note that since the ratio

$$\frac{t_\chi}{t_b} = (g\alpha\beta)^{1/2} l^2 \chi^{-1} , \quad (6b)$$

we have

$$\left(\frac{t_\chi}{t_b}\right)^2 = \frac{g\alpha\beta l^4}{\chi^2} = \sigma \, \mathrm{Ra} \equiv S , \quad (6c)$$

where Ra is the Rayleigh number ($= g\alpha\beta l^4/\nu\chi$). The dimensionless function S is related to the convective efficiency Γ by the relation, $\Sigma \equiv 2S/81$ (Cox & Giuli 1968),

$$\Gamma = \tfrac{1}{2}[(1 + \Sigma)^{1/2} - 1] . \quad (6d)$$

In the limit of high (low) efficiency, we have

$$\Sigma \gg 1 , \quad \Sigma \ll 1 , \quad (6e)$$

which we shall treat separately.

2.1. *High-Efficiency Regime $t_b \ll t_\chi$*

In this case, t_χ does not enter the problem. Physically, this regime corresponds to a situation whereby an eddy, a heat bubble, travels without losing heat via radiative processes. This can occur only if the buoyancy timescale is much shorter than the time t_χ it takes for the radiative losses to occur. In this case we can identify

$$\omega(k) \sim t_b^{-1} , \quad (6f)$$

so that equation (5b) gives

$$H_c \sim \left(\frac{\beta}{g\alpha}\right)^{1/2} \int E(k) dk . \quad (6g)$$

From Figure 2 we can observe that since in this limit the spectral function $E(k)$ is usually very wide, assuming the δ-function (4b) is bound to underestimate the integral in equation (6g). Thus, irrespectively of the detailed form of the full spectrum $E(k)$, one can conclude that

$$H_c(\mathrm{MLT}) < H_c(\text{full spectrum}) . \quad (7)$$

In the high-efficiency limit, the MLT underestimates the convective flux.

2.2. *Low-Efficiency Limit*

In this case, the derivation of $\omega(k)$ is somewhat more delicate since even though

$$t_\chi \ll t_b , \quad (8a)$$

the buoyancy timescale t_b must still be present since it represents the timescale of the source of instability. Thus, $\omega(k)$ must be a combination of both t_χ and t_b. The simplest expression consists of reducing the high-efficiency limit (6f) by the ratio t_χ/t_b. We thus write

$$\omega(k) \sim t_b^{-1}\left(\frac{t_\chi}{t_b}\right) \sim \frac{g\alpha\beta}{\chi k^2} . \quad (8b)$$

The growth rate decreases like k^{-2} (Fig. 3). If one uses a full spectrum $E(k)$, the k^{-2} dependence gets weighted appropriately. However, if one uses an MLT δ-function peaked at the largest eddy, as in the high-efficiency limit, one is actually weighing the low k region more than one should, with the consequence that one overestimates the convective

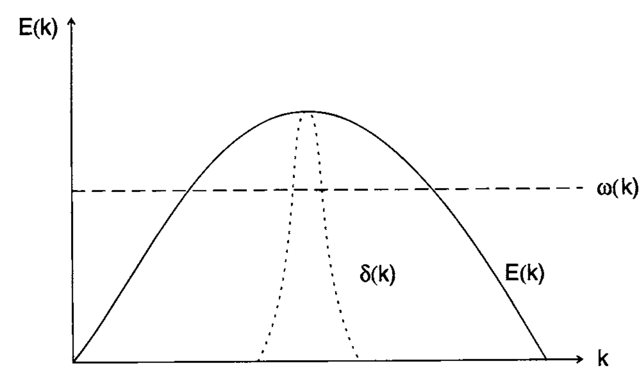

FIG. 2.—The energy spectrum $E(k)$ must be multiplied by the rate of energy input $\omega(k)$ in order to obtain the convective flux. In the case of highly efficient convection, $\omega(k) \sim$ constant and so the use of a δ-function approximation leads to underestimate the total kinetic energy and thus the flux.

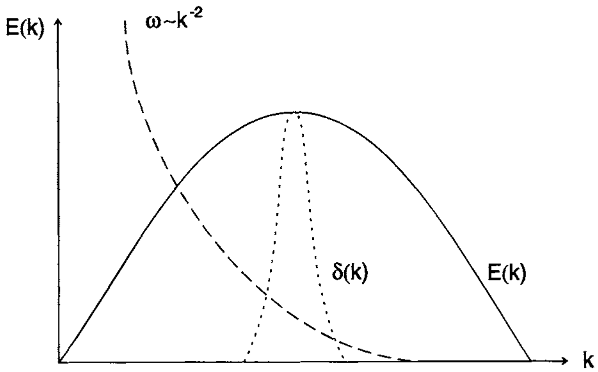

FIG. 3.—In the case of inefficient convection, $\omega(k) \sim k^{-2}$, the δ-function approximation peaked at the same k_0 as in the case of Fig. 2, gives an overestimate of the convective flux.

flux. Thus, on general grounds, one can conclude that

$$H_c(\text{MLT}) > H_c(\text{full spectrum}) . \tag{9}$$

In the low efficiency limit, the MLT overestimates the convective flux.

For completeness, we recall the MLT form of F_c (Gough & Weiss 1976),

$$H_c = \chi\beta\Phi , \tag{10a}$$

$$\Phi(\text{MLT}) = \tfrac{1}{2}a_0\Sigma^{-1}[(1+\Sigma)^{1/2} - 1]^3 , \tag{10b}$$

where ($a_0 = 9/4$) and

$$\Sigma \equiv \frac{2}{81} S , \quad S = \frac{g\alpha\beta l^4}{\chi^2} = 162 A^2(\nabla - \nabla_{\text{ad}}) , \tag{10c}$$

$$A \equiv \frac{l^2}{9\chi}\left(\frac{g}{2H_p}\right)^{1/2} , \quad \chi \equiv \frac{K}{c_p\rho} , \quad K = \frac{4acT^3}{3\kappa\rho} , \tag{10d}$$

where l is the mixing length, κ is the opacity, and all the other symbols have their usual meaning. We also recall that equations (6f) and (8b) are indeed the limiting cases of the general expression for the convective growth rate (Yamaguchi 1963; CM, eq. [18])

$$2\omega(k) = -\chi k^2(1+\sigma) + [\chi^2 k^4(1-\sigma)^2 + 4g\alpha\beta\tau(x)]^{1/2} , \tag{11}$$

where $\tau(x) = x(1+x)^{-1}$ with $x = k_h^2/k_z^2$, where k_h is the horizontal wavenumber.

3. DERIVATION OF THE MLT FLUX AS A ONE-EDDY MODEL

In this section, we justify the statement that equation (5b), together with equations (4b) and (11), implies the MLT expression (10b). We begin by considering that at any wavenumber k, the energy input from all the eddies smaller than k^{-1} is given by (see eq. [5c])

$$\int_{k_0}^{k} 2[\omega(k) + \nu k^2]E(k)dk . \tag{12a}$$

Since we must have a balance, the nonlinear interactions must distribute the energy (12a) to all the eddies larger than k^{-1}. Thus, we must equate equation (12a) to the "transfer function" $T(k)$, which represents the effect of the nonlinear interactions. The most familiar model for $T(k)$ is the one originally proposed by Heisenberg (Batchelor 1971) and recovered by all subsequent more complete turbulence models (Lesieur 1990). Its form is

$$T(k) = 2[\nu + \nu_t(k)]\int_{k_0}^{k} k^2 E(k)dk , \tag{12b}$$

which may be viewed as the extension of the molecular dissipation term,

$$2\nu\int_{k_0}^{k} k^2 E(k)dk , \tag{12c}$$

to include a k-dependent turbulent viscosity $\nu_t(k)$,

$$\nu \to \nu + \nu_t(k) . \tag{12d}$$

Several models for $\nu_t(k)$ have been derived over the years (Howells 1960; Canuto, Goldman, & Chasnov 1988; Lesieur 1990; Canuto & Dubovikov 1996a, b, c), and the general form is

$$\nu_t^2 = \nu^2 + \gamma\int_k^\infty E(p)p^{-2}dp . \tag{12e}$$

The fact that $\nu_t(k)$ is contributed by all eddies smaller than k^{-1} is an indication that the major contribution comes from the ultraviolet region while the infrared region is contributing to the energy input (eq. [12a]). The value of γ is $\frac{2}{5}$.

Equating equation (12a) to (12b), we have the highly nonlinear equation for $E(k)$,

$$\int_{k_0}^{k}[\omega(k) + \nu k^2]E(k)dk = [\nu + \nu_t(k)]\int_{k_0}^{k} k^2 E(k)dk . \tag{12f}$$

Using equation (4b), equation (12e) can easily be solved with the result

$$E_0 = \gamma^{-1}\omega_0^2 k_0^{-3} . \tag{12g}$$

Using equations (5b), (11), and $g\alpha H_c = \epsilon = g\alpha\overline{w\theta} = g\alpha\beta\chi\Phi$, we derive, after some algebra,

$$\Phi = C_X\Sigma^{-1}[(1+\Sigma)^{1/2} - 1]^3 , \quad C_X = \gamma^{-1}x(1+x)^{-1} , \tag{13}$$

which proves that the MLT expression (10b) can indeed be derived from a turbulence model under the one-eddy approximation (4b).

4. USE OF THE KOLMOGOROV SPECTRUM FOR $E(k)$

After the one-eddy MLT model and before using a full turbulence model to compute the kinetic energy spectrum $E(k)$, the next step is to use the Kolmogorov spectrum,

$$E(k) = \text{Ko}\,\epsilon^{2/3}k^{-5/3} , \tag{14a}$$

where Ko is the Kolmogorov constant whose latest determination yields (Praskovsky & Oncley 1994)

$$1.59 < \text{Ko} < 1.88 . \tag{14b}$$

As shown in Figure 1, a low-viscosity system (like stellar interiors), is dominated for a large fraction of the k spectrum by a Kolmogorov inertial regime, and thus equation (14a) is a good intermediate step between equation (4b) and the more complete model to be discussed in the next section.

Inserting equations (11) and (14a) into (5b), we obtain after some algebra ($\nu \to 0$)

$$\Phi = \left(\frac{3}{4}\text{Ko}\right)^3 \frac{4x}{1+x} I^3 , \tag{14c}$$

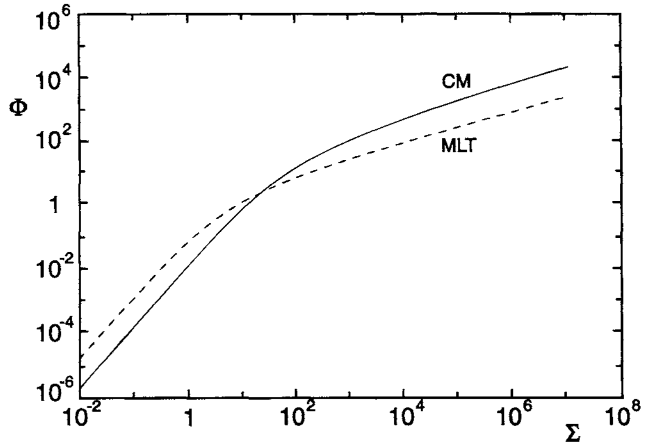

FIG. 4.—The convective flux of the CM model vs. the MLT. As one can see, the CM model does provide both a higher flux at high S and a lower flux at low S, as suggested by the arguments that led to Figs. 2 and 3.

where the dimensionless integral I is given by

$$I \equiv \int_a^\infty dt[(1 + t^{-3})^{1/2} - 1] , \quad a \equiv \Sigma^{-1/3} . \tag{14d}$$

Integration by parts in equation (14d) allows us to rewrite equation (14c) in the form

$$\Phi = \left(\frac{3}{4}\,\mathrm{Ko}\right)^3 \frac{4x}{1+x}\,\Sigma^{-1}[(1+\Sigma)^{1/2} - 1]^3 \phi^3 , \tag{14e}$$

which exhibits the structure of equation (13) but with the addition of the new function ϕ given by

$$\phi \equiv \tfrac{3}{2}\Sigma[(1+\Sigma)^{1/2} - 1]^{-1} \int_1^\infty t^{-3}\,dt(1 + \Sigma t^{-3})^{-1/2} - 1 . \tag{14f}$$

Using equation (13), we now have

$$\frac{\Phi}{\Phi(\mathrm{MLT})} = \frac{8}{5}\left(\frac{3}{4}\,\mathrm{Ko}\right)^3 \phi^3 . \tag{14g}$$

The presence of ϕ^3 causes Φ to be larger (smaller) than the MLT flux. In fact, since

$$\Sigma \gg 1 : \phi^3 \to 8 , \quad \Sigma \ll 1 : \phi^3 \to \tfrac{1}{8} , \tag{14h}$$

we derive

$$\Sigma \gg 1 : \frac{\Phi}{\Phi(\mathrm{MLT})} = 18\left(\frac{\mathrm{Ko}}{1.5}\right)^3 , \tag{14i}$$

$$\Sigma \ll 1 : \frac{\Phi}{\Phi(\mathrm{MLT})} = 0.3\left(\frac{\mathrm{Ko}}{1.5}\right)^3 . \tag{14j}$$

As we shall see, the more complete model confirms the general results, equations (14i)–(14j) (in the $\Sigma \gg 1$ regime, the factor is 11 rather than 18, while in the $\Sigma \ll 1$ regime, the factor is 0.1 rather than 0.3). Numerical evaluation of equation (14f) yields the following results: for $\Sigma = 10^{-2}, 10^{-1}$, 1, 10, 10^2, 10^3, 10^4, we have $\phi^3 = 0.125$, 0.13, 0.172, 0.387, 0.974, 1.964, 3.158, respectively.

5. FULL TURBULENT SPECTRUM: THE CM AND CGM MODELS AND ASTROPHYSICAL TESTS

Recently, two attempts have been made to bypass the MLT one-eddy approximation and the Kolmogorov law by solving a full turbulence model (CM; Canuto, Goldman, & Mazzitelli 1996 [hereafter CGM]) using a description of the nonlinear interactions more complete than the Heisenberg model (12b). As discussed in detail in Appendix B of CM, the transfer $T(k)$ is contributed by four terms of which equation (12b) is just the first (specifically, eq. [B.9] of CM). The CM Φ is

$$\Phi(\mathrm{CM}) = a_1 \Sigma^m[(1 + a_2\Sigma)^n - 1]^p \tag{15a}$$

with: $a_1 = 24.868$, $a_2 = 9.76610^{-2}$, $m = 0.14972$, $n = 0.18931$, $np + m = \frac{1}{2}$. In Figure 4, we show $\Phi(\mathrm{CM})/\Phi(\mathrm{MLT})$ versus Σ. The newest CGM flux Φ is given by

$$\Phi(\mathrm{CGM}) = aS^m[(1 + bS)^n - 1]^p \times [1 + cS^q(1 + dS^r)^{-1} + eS^s(1 + fS^t)^{-1}] , \tag{15b}$$

with $a = 10.8654$, $b = 0.00489073$, $c = 0.010871$, $d = 0.00301208$, $e = 0.000334441$, $f = 0.000125$ while the exponents are $m = 0.149888$, $n = 0.189238$, $p = 1.85011$, $q = 0.72$, $r = 0.92$, $s = 1.2$, and $t = 1.5$. The new models satisfy equations (7) and (9). In the two limits of interest, we have, using equation (10b):

$S \gg 1$:

$$\Phi(\mathrm{MLT}) = 0.176 S^{1/2} ,$$
$$\Phi(\mathrm{CM}) = 1.73 S^{1/2} ,$$
$$\Phi(\mathrm{CGM}) = 1.685\left(\frac{\mathrm{Ko}}{1.5}\right)^3 \left(\frac{\sigma_t}{0.72}\right)^{3/2} S^{1/2} , \tag{15c}$$

$S \ll 1$:

$$\Phi(\mathrm{MLT}) = 8.57 \times 10^{-5} S^2 ,$$
$$\Phi(\mathrm{CM}) = 0.942 \times 10^{-5} S^2 ,$$
$$\Phi(\mathrm{CGM}) = 2.65 \times 10^{-5}\left(\frac{\mathrm{Ko}}{1.5}\right)^3 S^2 . \tag{15d}$$

The CM model has been tested on several grounds comprising stellar structure and evolution, helioseismology and stellar atmospheres (D'Antona, Mazzitelli, & Gratton 1992; D'Antona & Mazzitelli 1994; Basu & Antia 1994; Baturin & Miranova 1995; Rosenthal et al. 1995; Stothers & Chin 1995; Monteiro, Christensen-Dalsgaard, & Thompson 1995; Canuto & Kupka 1996; Smalley 1995; Althaus & Benvenuto 1996). In all cases, the CM model performs better than the MLT. In addition, while the MLT has an adjustable parameter α, the CM has none, since the mixing length l is taken $l = z$, where z is the distance to the nearest "wall," where stratification changes from unstable to stable. In § 7 we shall present a new test of the MLT and CM-like models.

6. FLUX TILTING IN EFFICIENCY SPACE

Figure 4 shows that the new flux (eq. [15a]) corresponds to a "tilting" of the MLT curve rather than a simple increase or decrease. The convective flux is larger in the high-efficiency regime and smaller in the low-efficiency regime, an effect that cannot be achieved by changing α.

7. LABORATORY TEST OF MLT AND CM MODELS

In this section, we test both MLT and CM models against laboratory convection data. In Figure 5, the two plates at a distance D are kept at a fixed temperature difference $\delta T > 0$. Laboratory data (using helium) for the turbu-

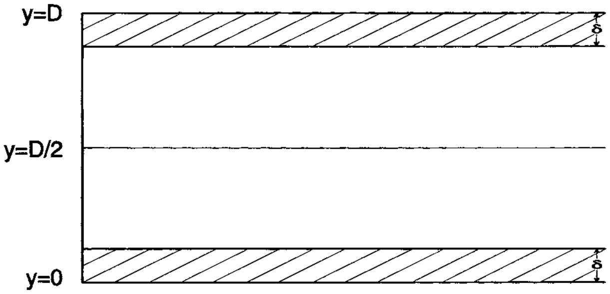

FIG. 5.—The geometry of laboratory convection. The temperature difference between the two plates is $\delta T > 0$; we have called δ the viscous layer.

lent heat transport are presented in the form (Castaing et al. 1989)

$$\mathrm{Nu} = c\,\mathrm{Ra}^n\,, \tag{16a}$$

where the Nusselt number Nu is the total heat flux H_t in units of $\chi\,\delta T/D$,

$$\mathrm{Nu} \equiv \frac{H_t}{\chi\,\delta T/D}\,, \tag{16b}$$

Ra is the experimentally known Rayleigh number,

$$\mathrm{Ra} = \frac{g\alpha\,\delta T}{\nu\chi}\,D^3\,, \tag{16c}$$

and H_t is given by (with $\beta = -\partial T/\partial z$),

$$H_t = H_{\rm cond} + H_{\rm conv} = \chi\beta + \chi\beta\Phi(S)\,, \tag{16d}$$

where the first term represents heat conduction and the second convection. The specific convection model enters through the function Φ. Since the latter depends on β, equation (16d) is a differential equation for the temperature whose solution yields the function $T(z)$. Since Φ depends on S and thus on l (eq. [6c]), the latter must be specified in terms of z. We suggest the form

$$l = q(z - \delta)\,, \quad y \geq \delta\,, \tag{17a}$$

where δ is the thickness of the viscous sublayer (see Fig. 5) for which we adopt the Kolmogorov expression

$$\delta = \left(\frac{\chi^3}{\epsilon}\right)^{1/4}\,, \tag{17b}$$

and where ϵ is computed at the middle of the layer (denoted by subscript M)

$$\epsilon \equiv \epsilon\Big|_{y=(1/2)D} = g\alpha\beta_M\,\chi\Phi_M\,. \tag{17c}$$

The quantity Φ_M is short for $\Phi(\Sigma_M)$ and

$$S_M = g\alpha\chi^{-2}\beta_M\Lambda_M^4\,, \quad \Lambda_M = q(\tfrac{1}{2}D - \delta) \approx \tfrac{1}{2}qD\,, \tag{17d}$$

since $D \gg \delta$; the parameter q will be taken equal to 2, but it is carried along for generality purposes. Since $H_t = \beta_{\rm max}\,\chi$, equation (16d) at mid-channel gives

$$\beta_{\rm max} = \beta_M(1 + \Phi_M)\,. \tag{18a}$$

Using equation (16b), equation (18a) becomes

$$\tfrac{2}{81}(\tfrac{1}{2}q)^4\ \mathrm{Ra\ Nu} = \Sigma_M(1 + \Phi_M)\,. \tag{18b}$$

Once a given model for convection—that is, a $\Phi(\Sigma)$—is chosen, the solution of equation (18b) yields

$$\Sigma_M \text{ versus Nu Ra}\,. \tag{18c}$$

After some manipulations, the thickness of the viscous layer δ (eq. [17b]) can be expressed as

$$\frac{\delta}{D} = (\mathrm{Ra\ Nu})^{-1/4}(1 + \Phi_M^{-1})^{1/4}\,. \tag{18d}$$

Next, we compute δT. We have by definition

$$T = \int_0^{(1/2)D} \beta(z)dz = \int_0^{\delta} \beta(z)dz + \int_{\delta}^{(1/2)D} \beta(z)dz\,, \tag{19a}$$

or, taking $\beta(z)$ constant in the first interval,

$$\delta T \equiv 2T = 2\beta_{\rm max}\,\delta + 2\int_{\delta}^{(1/2)D} \beta(z)dz\,. \tag{19b}$$

Substituting $H_t = \chi\beta_{\rm max} = \chi\,\delta T\,\mathrm{Nu}/D$ (eq. [16b]), and using the variable $x = \beta/\beta_{\rm max}$, equation (19b) becomes

$$\mathrm{Nu} = c(\mathrm{Ra})\ \mathrm{Ra}^{1/3}\,, \tag{20a}$$

where

$$2c^{3/4} \equiv (1 + \Phi_M^{-1} + I)^{-1}\,, \tag{20b}$$

where the integral I is given by

$$I \equiv (\mathrm{Ra\ Nu})^{1/4}D^{-1}\int_{\delta}^{(1/2)D} x(z)dz\,. \tag{20c}$$

From equation (16d) with $H_t = \chi\beta_{\rm max}$, we obtain $x(z) = [1 + \Phi(\Sigma)]^{-1}$. Next, change z/D to $Y \equiv (z - \delta)/D$ and then to Σ,

$$\Sigma = (\tfrac{2}{81}q^4)(1 + \Phi)^{-1}\ \mathrm{Ra\ Nu}\ Y^4\,. \tag{20d}$$

Finally,

$$I = \left(\frac{81}{2q^4}\right)^{1/4}\int_0^{\Sigma_M}[1 + \Phi(\Sigma)]^{-1}\,\frac{d}{d\Sigma}\,\{\Sigma[1 + \Phi(\Sigma)]\}^{1/4}\,d\Sigma \tag{20e}$$

To carry out the calculation, it is convenient to introduce the variable $\Psi \equiv \mathrm{NuRa}$, so that equation (20a) becomes

$$\Psi = c(\Psi)\ \mathrm{Ra}^{4/3}\,. \tag{21}$$

After choosing a convective model for $\Phi(\Sigma)$, one first solves equation (18b) to obtain Σ_M versus Ψ, which is then used in equation (20e); $I(\Psi)$ is then obtained and so is $c(\Psi)$. Equation (21) is then used to compute the corresponding Ra, which, once inserted in Ψ yields the corresponding Nu. The Nu versus Ra relationship thus follows. The results presented in Table 1 correspond to very large Ra, in which case $c(\mathrm{Ra}) \to c$. The Ra dependence of c leads to a power less

TABLE 1

THE NUSSELT NUMBER VERSUS RAYLEIGH NUMBER RELATIONSHIPS

Ra	Nu (exp)	CM	MLT
10^6	7.75	4.9	3.8
10^7	15	10.5	8.2
10^8	29	23	18
10^9	56.3	49	38.3
10^{10}	109	105.5	82.4

than $\frac{1}{3}$ in equation (20a). The data are from Castaing et al. (1989).

The CM model provides a better fit than the MLT. For an early comparison of MLT versus lab data, see Tooth & Gough (1988). Due to smallness of the Prandtl number σ and the fact that the quantity of interest is $S = \sigma$ Ra, the region of major interest for example in helioseismology is around $S \sim 1$ or less.

8. NONLOCAL, TIME-DEPENDENT, THEORY OF CONVECTION: THE REYNOLDS STRESS MODEL

The CM model is based on a two-point closure model, the EDQNM model, which has been extensively tested in many turbulent flows (Lesieur 1990) but which is limited to homogeneous and isotropic (or slightly anisotropic) flows. An approach that does not suffer from these limitations, and which is therefore more suited to study cases where inhomogeneities (nonlocal effects) dominate and have important consequences (the overshooting in stars), and which has been successfully used to describe types of turbulent flows, is the one-point, second-order closure model based on the Reynolds stress method. The dynamic equations for the second-order moments

$$h_i \equiv \overline{u_i \theta} \text{ (convective flux)}, \quad \overline{\theta^2} \text{ (temperature variance)},$$

$$\tau_{ij} \equiv \overline{u_i u_j} \text{ (Reynolds stress)},$$

$$K = \tfrac{1}{2}\tau_{ii} \text{ (turbulent kinetic energy)}$$

(Canuto 1992, 1993, 1994) are described below.

8.1. *General Equations*

The first is the *Reynold stress* equation, $\tau_{ij} \equiv \overline{u_i u_j}$:

$$\frac{D}{Dt}\tau_{ij} + D_f = \alpha h_i g_j^r + \alpha h_j g_i^r - \Pi_{ij} - \epsilon_{ij}, \tag{22a}$$

where the pressure-velocity correlation Π_{ij} (a third-order moment) is given by

$$\Pi_{ij} = 2c_4\tau^{-1}b_{ij} + c_5\alpha g_k(\delta_{ik}h_j + \delta_{jk}h_i - \tfrac{2}{3}\delta_{ij}h_k), \tag{22b}$$

$$b_{ij} = \tau_{ij} - \tfrac{2}{3}\delta_{ij}K, \tag{22c}$$

$$\epsilon_{ij} = \tfrac{2}{3}\epsilon\delta_{ij}. \tag{22d}$$

Here, g^r is the renormalized gravity given by

$$g_i^r = g_i + \frac{\partial}{\partial x_j}\tau_{ij}. \tag{22e}$$

The diffusion term D_f, a third-order moment representing nonlocality, will be discussed below. The *heat flux* equation is

$$\frac{D}{Dt}h_i + D_f = \tau_{ij}\beta_j^r + \alpha g_i^r\overline{\theta^2} - \Pi_i^\theta + \frac{1}{2}\chi\frac{\partial^2}{\partial x_j^2}h_i, \tag{23a}$$

where the pressure-temperature correlation Π_i^θ is given by

$$\Pi_i^\theta = 2c_6\tau^{-1}h_i + c_7\alpha g_i\overline{\theta^2}. \tag{23b}$$

Here, β_i^r is the renormalized superadiabatic gradient defined by

$$\beta_i \equiv -\frac{\partial T}{\partial x_i} - \frac{1}{c_p}g_i^r = -\frac{\partial T}{\partial x_i} - \frac{1}{c_p}\left(g_i + \frac{\partial}{\partial x_j}\tau_{ij}\right) \tag{23c}$$

which we shall discuss below.

The *temperature variance*, $\overline{\theta^2}$, is given by

$$\frac{D}{Dt}\overline{\theta^2} + D_f = 2h_i\beta_i^r - 2\epsilon_\theta + \chi\frac{\partial^2}{\partial x_j^2}\overline{\theta^2}. \tag{24}$$

The *hydrostatic equilibrium* equation ($P = P_{\rm gas} + P_{\rm rad}$) is

$$\frac{\partial P}{\partial x_i} = -g_i^r. \tag{25a}$$

The *mean temperature* equation is

$$\frac{D}{Dt}(c_p T + K) = -\frac{\partial}{\partial x_j}[F_j^{\rm rad} + F_j^{\rm conv} + F_j^{\rm KE}], \tag{25b}$$

where the radiative, convective, and kinetic fluxes are defined by ($c_p\rho\chi = K_{\rm rad}$)

$$F_j^{\rm rad} = -K_{\rm rad}\frac{\partial T}{\partial x_j}, \quad F_j^{\rm conv} = c_p\rho\overline{u_j\theta}, \quad F_j^{\rm KE} = \frac{1}{2}\rho\overline{q^2 u_j}. \tag{25c}$$

All the diffusion terms are defined and expressed in terms of the second-order moments by equations (71)–(76) of Canuto (1993). As one can see, these new expressions can easily absorb the renormalized parts of each g^r and thus, from now on, we shall call them D^r and restore the g's in the right-hand sides of eqs. (22a)–(23a).

8.2. *One-dimensional Case*

The *turbulent kinetic energy in the z-direction* is given by

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{w^2} + D_f^r = -c_4\tau^{-1}\left(\overline{w^2} - \frac{2}{3}K\right) + \left(1 - \frac{2}{3}c_5\right)g\alpha\overline{w\theta} - \frac{1}{3}\epsilon, \tag{26a}$$

where *turbulent kinetic energy* K is given by

$$\frac{\partial K}{\partial t} + D_f^r = g\alpha\overline{w\theta} - \epsilon, \tag{26b}$$

and *convective flux* $\overline{w\theta}$ is given by

$$\frac{\partial}{\partial t}\overline{w\theta} + D_f^r = \beta_r\overline{w^2} - 2c_6\tau^{-1}\overline{w\theta} + (1 - c_7)g\alpha\overline{\theta^2} + \frac{1}{2}\chi\frac{\partial^2}{\partial z^2}\overline{w\theta}, \tag{27}$$

where *temperature variance* $\overline{\theta^2}$ is given by

$$\frac{\partial\overline{\theta^2}}{\partial t} + D_f = 2\beta_r\overline{w\theta} + \chi\frac{\partial^2}{\partial z^2}\overline{\theta^2} - 2\epsilon_\theta. \tag{28}$$

The *hydrostatic equilibrium* is given

$$\frac{\partial}{\partial z}P = -g^r \tag{29a}$$

where the normalized gravity g^r and superadiabatic gradient β^r are given by

$$g^r = g\left(1 + \frac{1}{g}\frac{\partial}{\partial z}p_t\right), \tag{29b}$$

$$\beta^r = -\frac{\partial T}{\partial z} - \frac{g^r}{c_p} = -\frac{\partial T}{\partial z} - \frac{g}{c_p}\left(1 + \frac{1}{g}\frac{\partial}{\partial z}p_t\right). \tag{29c}$$

Some comments are in order here. The superadiabatic gradient (23c) is different from the one suggested in previous works. For example, Baker & Gough (1979) suggested (but did not derive) that the effect of the turbulent pressure can be accounted for by replacing P(gas + radiation) in the standard definition of β (their eq. [2.12]) with $P + p_t$, with the result

$$\beta = -\frac{\partial T}{\partial r} + \frac{1}{c_p}\frac{\partial P}{\partial r} \to \beta^r = -\frac{\partial T}{\partial r} + \frac{1}{c_p}\frac{\partial}{\partial r}(P + p_t) , \quad (30a)$$

which they write as (their eq. [2.17])

$$\beta^r = -\frac{\partial T}{\partial r} - \frac{g}{c_p}\frac{P}{P + p_t} . \quad (30b)$$

Neither expression for β^r is consistent with the Reynolds stress model (eqs. [23c] and [29c]). Cox & Giuli (1968, fn., p. 295), on the other hand, suggest the expression

$$\beta^r = -\frac{\partial T}{\partial z} - \frac{g}{c_p}\frac{P + p_t}{P}\left(1 + \frac{1}{g}\frac{\partial p_t}{\partial z}\right) , \quad (30c)$$

which is closer to our result, equation (29c).

Solution of the above equations, using the diffusion terms derived in C93, offers the most complete description of overshooting. In the case of the strongly convective planetary boundary layer, the above equations were solved and shown to give very good results (Canuto et al. 1994a).

8.3. *Local, Stationary Limit: Exact Solutions*

Next, we solve the above system of equations in the local, stationary limit

$$\frac{\partial}{\partial t} = 0 , \quad D_f \to 0 , \quad \frac{\partial^2}{\partial z^2} \to -\Lambda^{-2} . \quad (31a)$$

These solutions serve to clarify many of the properties of the system and as a check of the full numerical solutions. We begin with the following basic turbulence relations

$$\epsilon_\theta = 2c_2 \tau^{-1}\overline{\theta^2} , \quad \tau = 2K\epsilon^{-1} , \quad (31b)$$

$$\epsilon = c_\epsilon K^{3/2} l^{-1} , \quad c_\epsilon = \pi\left(\frac{2}{3\,\mathrm{Ko}}\right)^{3/2} . \quad (31c)$$

The first relation is the assumption that the dissipation timescale of potential energy is proportional to that of turbulent kinetic energy, which is valid as long as the stratification is unstable; the second relation is the definition of the timescale τ, while the third and fourth relations result from substituting the Kolmogorov law (14a) into equation (1a) and integrating from π/l to infinity.

Next, from equation (26c) we have that $\epsilon = g\alpha\overline{w\theta}$ and write $\epsilon = g\alpha\beta\chi\Phi$, so as to introduce the dimensionless function Φ. We derive

$$\tau K \chi^{-1} = 2c_\epsilon^{-4/3}(S\Phi)^{1/3} , \quad \tau^2 N^2 = 4c_\epsilon^{-4/3} S(S\Phi)^{-2/3} \quad (31d)$$

where

$$S = g\alpha\beta l^4 \chi^{-2} \quad (31e)$$

Solving equations (26)–(28), we obtain:

1. Turbulent kinetic energy K:

$$K = \frac{3}{2}\pi^{-2/3}\,\mathrm{Ko}\,(S\Phi)^{2/3}\left(\frac{\chi}{l}\right)^2 ; \quad (32a)$$

2. Turbulent kinetic energy in the z-direction:

$$\frac{1}{2}\overline{w^2} = \frac{1}{3}K\left[1 + \frac{2}{c_4}(1 - c_5)\right] ; \quad (32b)$$

3. Temperature variance $\overline{\theta^2}$:

$$\frac{1}{2}\overline{\theta^2} = (1 + 2c_2\,\mathrm{Pe})^{-1}\Phi\left(\frac{\Lambda}{l}\right)^2 (l\beta)^2 ; \quad (32c)$$

4. Potential energy, PE ($N^2 \equiv g\alpha\beta$):

$$\mathrm{PE} \equiv \frac{1}{2}\overline{\theta^2}(g\alpha)^2 N^{-2} = (1 + 2c_2\,\mathrm{Pe})^{-1} S\Phi\left(\frac{\Lambda}{l}\right)^2\left(\frac{\chi}{l}\right)^2 ; \quad (32d)$$

5. Velocity-temperature correlation (convective flux):

$$\overline{w\theta} = \beta\chi\Phi ; \quad (32e)$$

$$\Phi = \mathrm{Ko}^3\, C(\mathrm{Pe}) S^{1/2} ; \quad (32f)$$

$$C(\mathrm{Pe}) = \left(\frac{27}{\pi^4}\right)^{1/2}\left[1 + \frac{2}{c_4}(1 - c_5) + 3(1 - c_7)\,\mathrm{Pe} \times (1 + 2c_2\,\mathrm{Pe})^{-1}\right]^{3/2}\left(\frac{\mathrm{Pe}}{1 + 2c_6\,\mathrm{Pe}}\right)^{3/2} ; \quad (32g)$$

6. Peclet number Pe:

$$\mathrm{Pe} \equiv \frac{2\Lambda^2}{\tau\chi} = c_\epsilon^{2/3}\left(\frac{\Lambda}{l}\right)^2 (S\Phi)^{1/3} . \quad (32h)$$

Equation (32f) is actually a cubic equation in Pe that can be solved to obtain the desired Φ versus S relationship. However, the form (32f) is physically more appealing for it exhibits some interesting features:

1. The convective flux scales like the cube of the Kolmogorov constant, a dependence already found in two previous models.
2. In the high-efficiency regime, $\chi \to 0$, Pe $\gg 1$, and we have

$$\Phi = \mathrm{Ko}^3\, C S^{1/2} , \quad (32i)$$

where the constant C is

$$C \equiv C(\infty) = \left(\frac{27}{\pi^4}\right)^{1/2}\left[1 + \frac{2}{c_4}(1 - c_5) + \frac{3}{2c_2}(1 - c_7)\right]^{3/2}\left(\frac{1}{2c_6}\right)^{3/2} , \quad (32j)$$

so that $\Phi \sim S^{1/2}$, as predicted by both MLT and the CM–CGM models.

3. In the low-efficiency regime, Pe < 1, we derive

$$\Phi = \mathrm{Ko}^3\left(\frac{\Lambda}{l}\right)^6 C_0 S^2 , \quad (32k)$$

with

$$C_0 = \frac{8}{\pi^2}\left[1 + \frac{2}{c_4}(1 - c_5)\right]^3 . \quad (32l)$$

The flux scales like Ko^3 and S^2, in accordance with the CM–CGM models. The S^2 dependence is also found in the MLT. To evaluate the numerical coefficient in front of S^2, we use the CM and CGM models (eqs. [15c]) and obtain $\Lambda/l = 9 \times 10^{-2}$.

4. Finally, we compute the constants C's in equations (32j) and (32k). Using the values (Canuto 1992, eq. [44d])

$$c_2 = 2 \ , \quad c_4 = 1.75 \ , \quad c_5 = 0.3 \ , \quad c_6 = 3.75 \ , \quad c_7 = 0.25 \ , \tag{33a}$$

we obtain

$$C = 9.3 \times 10^{-2} \ , \quad C_0 = 4.73 \ , \tag{33b}$$

so that, using equation (14b), we have (eq. [32i]):

$$0.4 \le \mathrm{Ko}^3 \, C \le 0.64 \ , \tag{33c}$$

$$\Phi = (0.4\text{–}0.64) S^{1/2} \ , \tag{33d}$$

to be contrasted with (eq. [15c]):

$$\Phi(\mathrm{MLT}) = 0.176 S^{1/2} \ . \tag{33e}$$

The new flux is (2.2–4) times larger than the MLT value. To recover the MLT value from the Reynolds stress model, one must take Ko = 1.23, which is totally outside the experimental range (eq. [14b]).

Thus, quite a different methodology leads us to the same conclusion that the MLT underestimates the convective flux.

9. TURBULENT VISCOSITY

In turbulence studies, the concept of turbulent viscosity ν_t and/or turbulent conductivity has proven very useful. Here, we show that the traditional definitions borrowed from flows where radiative losses play no role cannot be used in stars for they lead to inconsistencies. A new expression is then suggested. We begin by writing

$$\overline{w\theta} = \chi_t \beta_r = \beta_r \chi \Phi \tag{34a}$$

and inquire whether the well-known Kolmogorov–Prandtl expression for χ_t,

$$\chi_t \sim vl \sim K^{1/2} l = \sigma_t^{-1} C_\mu \frac{K^2}{\epsilon} \ , \tag{34b}$$

is valid for both efficient and inefficient convection. Here, σ_t is the turbulent Prandtl number (~0.5–0.72) and the coefficient C_μ is traditionally taken to be 0.09–0.11 in neutrally stratified flows. Substituting the second relation of equation (34b) into equation (34a), we derive

$$\Phi = \left(\frac{C_\mu}{\sigma_t}\right)^{1/2} \frac{K}{\chi N} \ . \tag{34c}$$

With the K given by equation (32a), we obtain

$$\Phi = a S^{1/2} \ , \tag{34d}$$

$$a \equiv \tfrac{27}{8} \mathrm{Ko}^3 \, \pi^{-2} (C_\mu/\sigma_t)^{3/2} \ . \tag{34e}$$

Equation (34d) does not coincide with the general expression (32f), only with its high-efficiency limit, equation (32i). Thus, equation (34b) is incomplete and a Pe-dependent function is required to obtain a result consistent with equation (32f). This is achieved by changing the second relation of equation (34b) to

$$\chi_t = \frac{K^2}{\epsilon} \psi(\mathrm{Pe}) \tag{34f}$$

with

$$\psi(\mathrm{Pe}) = \left[\frac{8\pi^2}{27} C(\mathrm{Pe})\right]^{2/3} \ , \tag{34g}$$

where C(Pe) is given by equation (32g). Equation (34f) is the new expression for the turbulent conductivity that we suggest should be used. Of course, equation (34f) coincides with equation (32e) that

$$\chi_t = \chi \Phi \ . \tag{34h}$$

10. NEW FLUX CONSERVATION EQUATION

The mean temperature profile is derived upon solving equation (25b). In general, it is advisable not to take the stationary case from the very beginning but rather solve equation (25b) with $D/Dt \to \partial/\partial t$ as it corresponds to the case of no mean flow. In the stationary case, equation (25b) becomes the flux conservation law (Canuto 1993)

$$F_{\mathrm{rad}} + F_{\mathrm{conv}} + F_{\mathrm{KE}} = F_{\mathrm{tot}} \ , \tag{35a}$$

where

$$2F_{\mathrm{KE}} = g\alpha\tau E_1 \frac{\partial}{\partial z} \overline{w\theta} + E_2 \frac{\partial}{\partial z} \overline{w^2} + (g\alpha\tau)^2 E_3 \frac{\partial}{\partial z} \overline{\theta^2} + 2E_4 \frac{\partial}{\partial z} K \ . \tag{36a}$$

The four functions E_k have the form

$$E_k = E_{k_1} \tau \overline{w^2} + E_{k_2} g\alpha\tau^2 \overline{w\theta} \ , \tag{36b}$$

while the dimensionless functions $E_{k_{1,2}}$ are given in Appendix B of Canuto (1993). The variable τ can be expressed via equations (31d). The second-order moments are given by equations (32a)–(32h). Contrary to the standard case, equation (35a) is no longer an algebraic relation but a differential equation. It would be interesting to see what results one obtains in the case of the sun where one can constraint the resulting $T(z)$ versus z profile using helioseismological data. The physical reason why we suggest this new method is because in the middle of the convective zone (CZ), F_{KE} is probably rather small due to the flatness of the second-order moments. However, when approaching the boundaries of the CZ, the second-order moments exhibit significant curvatures yielding a nonzero F_{KE}, which may even be larger than F_{conv}.

11. TURBULENT PRESSURE

Before we use the Reynolds stress model to compute the turbulent pressure p_t, we present some physical arguments as to its expected form. Since p_t/ρ has the dimension of energy, we need a length and a timescale. The first is l, while the second depends on whether convection is efficient or inefficient. In the case of efficient convection, $\mathrm{Pe} \gg 1$, the timescale is given by equation (6g) and using the first relation of equation (6a), we have

$$p_t \sim g\alpha\beta l^2 \sim \left(\frac{g\alpha\beta l^4}{\chi^2}\right)\left(\frac{\chi}{l}\right)^2 \sim S\left(\frac{\chi}{l}\right)^2 \ . \tag{37a}$$

The first relation shows that χ does not enter the problem, as expected. The last expression is constructed to exhibit the fact that $(\chi/l)^2$ is the natural unit for p_t. If we take the thermal pressure to be $p_{\rm th} \sim gH_p$, we have

$$\frac{p_t}{p_{\rm th}} \sim \left(\frac{l}{H_p}\right)^2 (\nabla - \nabla_{\rm ad}) , \quad (37b)$$

and since in this case $\nabla - \nabla_{\rm ad} \sim O(10^{-8})$, one can expect a small contribution from p_t.

In the case of inefficient convection, Pe $\ll 1$, the timescale is given by equation (8b) and so we expect that

$$p_t \sim \left(\frac{g\alpha\beta l^2}{\chi}\right)^2 l^2 \sim S^2\left(\frac{\chi}{l}\right)^2 . \quad (38)$$

In this case

$$\frac{p_t}{p_{\rm th}} \sim \left(\frac{l}{H_p}\right)^2 (\nabla - \nabla_{\rm ad}) S . \quad (39)$$

Contrary to the previous case, here $\nabla - \nabla_{\rm ad} \sim O(1)$ and $S \sim O(1)$, so that even if l is smaller than the previous case ($l \sim z$, where z is the distance to the nearest wall), the contribution from p_t is expected to be nonnegligible. A complete model must yield an expression that reduces to equations (37a) and (38) in the two limits. From equation (32b), we derive

$$\rho^{-1} p_t = \frac{1}{2} \, {\rm Ko} \, C_0^{1/3} (S\Phi)^{2/3} \left(\frac{\chi}{l}\right)^2 , \quad (40)$$

where Φ and C_0 are given by equations (32f) and (32l). It is easy to check that in the two limits, equations (32i) and (32k), equation (40) reduces to equations (37a) and (38), respectively. Equation (40) is the expression for the turbulent pressure within the one-point closure, Reynolds stress model.

In the case of a two-point closure model which yields the energy spectrum $E(k)$, the turbulent pressure is computed via the expression (Canuto et al. 1996b; CGM model)

$$\rho^{-1} p_t = (8\pi^2)^{-1/2} \left[\iiint dk\, dk' \sin\phi E(k) E(k') |\boldsymbol{k} - \boldsymbol{k}'|^{-4}\right]^{1/2} \quad (41)$$

where ϕ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. In the case of the new CGM model, the result is

$$S \gg 1: \ \rho^{-1} p_t = 0.4689 \left(\frac{\rm Ko}{1.5}\right)^3 \left(\frac{\sigma_t}{0.72}\right) S \left(\frac{\chi}{l}\right)^2 , \quad (42a)$$

$$S \ll 1: \ \rho^{-1} p_t = 5.3408 \times 10^{-4} \left(\frac{\rm Ko}{1.5}\right)^3 S^2 \left(\frac{\chi}{l}\right)^2 . \quad (42b)$$

The formula for arbitrary S values can be found in the CGM paper. As one can notice, the one-point closure Reynolds stress model and the two-point closure CGM model yield very similar results.

12. RADIATIVE LAYER ON THE CONVECTIVE FLUX

We consider the effect that a convectively stable layer has on the convective flux F_c. The physical implications of such a layer depend on the values of the temperature gradients in the two regions. In the absence of a stable layer, it is known that convection sets in when the Rayleigh number is larger than the critical value $({\rm Ra})_c = 675.5$. The presence of a stable layer can either facilitate the onset of convection (if ${\rm Ra}_s < 6300$, where ${\rm Ra}_s$ is the Rayleigh number of the stable layer) or hinder it ${\rm Ra}_s > 6300$. In general, when S_u (where the subscript u means unstable) is very large ($\geq 10^6$), the effect of the stable layer is small; on the other hand, for $S_u \leq 10^4$, the stable layer affects the growth rates and thus the convective flux since $F_c \sim \omega^3$. We define the geometry as follows

$$0 \leq z \leq D, \text{ convectively unstable layer, } \nabla - \nabla_{\rm ad} > 0 , \quad (43a)$$

$$D \leq z \leq \infty, \text{ convectively stable layer, } \nabla - \nabla_{\rm ad} < 0 . \quad (43b)$$

At the boundary $z = D$, the velocity w is no longer zero, as in the derivation of equation (11); rather, at $z = D$, the velocity must be continuous while it vanishes at $z = 0$ and $z = \infty$. At $z = D$, one imposes that the functions

$$w(z) , \quad w^{(n)}(z) , \quad n = 1, \ldots, 5 , \quad (43c)$$

where (n) denotes the nth derivative, be continuous. The linear stability analysis can be carried out exactly with the following result for the growth rate $\omega(k)$. Consider equation (11). For $\sigma = 0$ and with $x = k_h^2/k_z^2$ and $k_z = n\pi$, $n = 0$, 1, ..., we have

$$2\omega(k) = -\chi k^2 + [\chi^2 k^4 + 4g\alpha\beta(1 - n^2\pi^2/k^2)]^{1/2} \quad (44)$$

Savolainen, Canuto, & Schilling (1992, hereafter SCS) have shown that the growth rate corresponding to equations (43a)–(43b) can be cast in the form (44), with $kD = q$,

$$2(\chi/D^2)\omega(k) = -q^2 + [q^4 + 4S(1 - n^2\pi^2/q^2)]^{1/2} \quad (45)$$

provided both n and S are now functions of the two variables

$$S_u = g\alpha|\beta_u| D^4 \chi^{-2} , \quad S_s = g\alpha|\beta_s| D^4 \chi^{-2} , \quad (46a)$$

$$\beta \equiv -\frac{\partial T}{\partial z} + \left(\frac{\partial T}{\partial z}\right)_{\rm ad} \quad (46b)$$

and the subscripts u and s refer to the unstable convective region (CZ) and to the stable one (radiative region). The parameterization suggested by SCS is

$$S \equiv S_u[1 + F \exp(-a_6 q_\perp)] , \quad n \equiv (a_1 + a_2 x^{0.55})^{-1} + a_3 , \quad (47a)$$

$$F = (a_4 + a_5 x^{-0.85})^{-1} , \quad x = |\beta_u|/|\beta_s| .$$

Moreover,

$$a_i = \sum_{j=1}^{4} a_{ij} t^{j-1} , \quad i = 1, \ldots, 5 \quad t = \log_{10} S_u , \quad (47b)$$

$$a_6 = (1.5 + 0.7x^{-0.6})^{-1} + 0.2 . \quad (47c)$$

The coefficients a_{ij} are given by equation (14) of SCS.

Following a procedure analogous to the one used in § 4, we derive the following results. In lieu of equations (14e)–(14f), we now have

$$\Phi = (\tfrac{3}{4}\,{\rm Ko})^3 \Sigma_*^{-1} [(1 + A)^{1/2} - 1]^3 \phi^3 , \quad (48a)$$

where

$$\phi = \tfrac{3}{2}[(1 + A)^{1/2} - 1]^{-1} I - 1 , \quad (48b)$$

$$I \equiv \int_1^\infty x^{-3} dx [1 + x^{-3} A(x)]^{-1/2} A(x) \left[1 - \frac{1}{3} \frac{x}{A(x)} \frac{d}{dx} A(x)\right], \quad (48c)$$

$$A(x) = 4\Sigma_* \left[1 - \frac{n^2\pi^2}{(k_0 D)^2} x^{-2}\right][1 + F \exp(-a_6 q_\perp)] , \tag{48d}$$

$$A \equiv A(x = 1) , \quad \Sigma_* \equiv g\alpha\beta_u \chi^{-2} k_0^{-4} . \tag{48e}$$

The remaining symbols have already been defined. Even though the integration in equation (48c) can in principle be carried out analytically, the result is a generalized hypergeometric function that is not easier to handle (numerically) than the integral itself. The distance D is not known a priori but it must be computed self-consistently through an iterative procedure.

13. FUTURE RESEARCH

Even though from the point of view of turbulence modeling and performance versus a variety of data, the CM model is quite satisfactory, we view it as the first of a hierarchy of post-MLT models that ought to be constructed and tested. Specifically:

1. *The CM model.*—In this model one employs the linear growth rate to quantify the rate of energy input. Due to the long lifetimes of the large eddies, this is not a bad approximation but it must be improved. In Canuto et al. (1988), it was shown that a self-consistent method exists whereby the growth rate is computed as a function of the turbulence itself, and one may expect that such an effect will lower the convective flux, mostly in the low S regime. In fact, one such model has recently been constructed (Canuto et al. 1996b) with the following result: the CM results remain unchanged in the high S regime while at low S, the new flux is 3 times smaller than the MLT value, rather than 10 times smaller, as predicted by the CM model. The comparison is, however, not quite complete since in the new model the nonlinear interactions are not treated with the same model as in CM. Thus, the self-consistent growth rate ought to be computed not only with the simplified representation of the nonlinear interactions used by CGM but also with the more complete model used in CM.

2. *Two-point closure models.*—The challenge is to do away entirely with the growth rates. This can be accomplished if one solves the dynamic equations for the relevant fluxes

$$\overline{w\theta} = \int H(k) dk , \tag{49}$$

$$\tfrac{1}{2}\overline{\theta^2} = \int G(k) dk , \tag{50}$$

$$\tfrac{1}{2} v_t^2 = \int E(k) dk . \tag{51}$$

The equations in question are (Yamaguchi 1963; Canuto et al. 1988)

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k) = g\alpha H(k) + T_E(k) , \tag{52a}$$

$$\left(\frac{\partial}{\partial t} + 2\chi k^2\right) G(k) = \beta H(k) + T_G(k) , \tag{52b}$$

$$\left[\frac{\partial}{\partial t} + (\nu + \chi) k^2\right] H(k) = \frac{1}{2}\beta\tau_1 E(k) + \frac{1}{2} g\alpha\tau_2 G(k) + T_H(k) . \tag{52c}$$

Clearly, at this stage one has to specify the transfer terms T to represent the effect of the nonlinear interactions. One may employ for the form used in CM, equation (21):

$$T_E(k) = \int_\Delta \int dp\, dq\, E(q)[k^2 E(p) - p^2 E(k)] a(p, q, k) \theta(k, p, q) , \tag{53a}$$

$$T_G(k) = \int_\Delta \int dp\, dq\, E(q)[k^2 G(p) - p^2 G(k)] b(p, q, k) \theta(k, p, q) , \tag{53b}$$

$$T_H(k) = -k^2[\nu_t(k) + \chi_t(k)] , \tag{53c}$$

where the functions a and b are geometrical factors, denote the turbulence timescales, $\chi_t(k) = \sigma_t \nu_t(k)$, where σ_t is the turbulent Prandtl number, and Δ assures momentum conservation, $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{k}$.

3. *One-point closure models.*—Two-point closure models can account for the full energy spectrum (e.g., the CM model) but cannot account for the effects of inhomogeneities and anisotropy. Since the latter are important for describing overshooting and the coupling of pulsation to convection, one must resort to another turbulence model. The most appropriate is the Reynolds stress model, which, in its incompressible version, has not yet been fully employed in either case. Since such a model has been successfully used in describing the strongly convective planetary boundary layer (Canuto et al. 1994a) and the interaction between shear, vorticity, and buoyancy (Canuto et al. 1994b), the model ought to be used in the context of helioseismology.

4. *Effect of radiative (stable) layer.*—It is known that the presence of a stable layer on top of an unstable layer can either enhance or hinder the convective instability, depending on the ratio of the temperature gradients in the two regions. No comprehensive treatment of this phenomenon has yet been given. In § 12, we have sketched what we consider the first step in that direction. Rather than using an MLT approach, we have worked out the expression for the convective flux in the case of an energy spectrum represented by a Kolmogorov law since we have previously seen that it is a good representation of the more complete energy spectrum. The model can be easily incorporated in a stellar structure code and its effects tested.

14. CONCLUSIONS

It is generally agreed that a theory cannot be proved, only disproved, and that a theory that cannot be falsified is not a physical theory. In that sense, MLT and the CM–CGM models differ quite substantially. The MLT can (almost) never be falsified since the parameter α makes MLT adaptable to almost all circumstances, if one of course is not uncomfortable with using different α's in different astrophysical situations. Among other reasons (e.g., lack of alternatives), this illusory resiliency of the MLT may have been at the root of its longevity in spite of the fact that in fields other than astrophysics where turbulent convection is important, the MLT has been forsaken in favor of more predictive models. The flexibility provided by α has regrettably been misconstrued as a sign of physical robustness, thus ultimately acting as a disincentive toward the construction of better models. In that sense, the MLT may have unwittingly been an impediment to further progress. In fact, turbulent heat transport is the only process in stellar structure studies that has not been changed since the MLT was proposed some 30 years ago. On the other hand, the CM–CGM models are falsifiable since the predictions are unequivocal: the absence of adjustable parameters prevents us from being mislead into believing that these models will last forever. The primary goal of the CM–CGM models was to remove the one-eddy MLT approximation and to account for the full eddy spectrum. The CM–CGM models are of course perfectible. For this reason, we have discussed a set of improved models that ought to be constructed as successors of the CM–CGM models. The one-point closure model based on the Reynolds stress procedure is the best example.

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